Matrix inverse

- Determinants and non-singularity

The determinant $|A|$ of a 2x2 matrix, called a second-order determinant, is derived by taking the product of the two elements on he principal diagonal and subtracting from it the product of the two elements of the principal diagonal. Given a general 2x2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The determinant is $|A|$ $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

The determinant is a single number or scalar and is found only for square matrices. If the determinant of a matrix is equal to zero, the determinant is said to vanish and the matrix is termed singular. A singular matrix is one in which there exists linear dependence between at least two rows or columns. If $|A| \neq 0$, matrix A is nonsingular and all its rows and columns are linearly independent

The rank ρ of a matrix is defined as the maximum number of linearly independent rows or columns in the matrix.

- Third-Order determinants

The determinant of a 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \mathrm{a}_{32} & \mathrm{a}_{33} \end{bmatrix}$$

Is called a third-order determinant and is the summation of three products. To derive the three products:

1. Take the first element of the first row, a_{11}, and mentally delete the row and column in which it appears. The multiply a11 by the determinant of the remaining elements

2 take the second element of the first row, a_{12}, and mentally delete the row and column in which it appears. Then multiply a_{12} by –1 times the determinant of the remaining elements

3 Take the third element of the first row, a_{13}, and mentally delete the row and column in which it appears . Then multiply a_{13} by the determinant of the remaining elements

Then the calculations of the determinant are as follows:

$$
\begin{aligned}
|A| &= a_{11}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + a_{12}(-1)\begin{bmatrix} a_{21} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + a_{13}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= \text{a } s\text{calar}
\end{aligned}
$$

Minors and cofactors

The elements of a matrix remaining after the deletion process we described before from a subdeterminant of the matrix is called a minors. Thus, a minor $|M_{ij}|$ is the determinant of the submatrix formed by deleting the *ith* row and *jth* column of the matrix, Using the matrix from the above

$$
|M_{11}| = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \qquad |M_{12}| = \begin{bmatrix} a_{21} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \qquad |\mathrm{M}_{13}| = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}
$$

Where $|M_{11}|$ is the minor of a_{11}, $|M_{12}|$ is the minor of a_{12} and $|M_{13}|$ is the minor of a_{13} Thus, the determinant can be rewritten

$$
|A| = a_{11}|M_{11}| + a_{12}(-1)|M_{12}| + a_{13}|M_{13}|
$$

A cofactor $|C_{ij}|$ is a minor with a prescribed sign. The rule for the sign of a cofactor is

$$
|C_{ij}| = (-1)^{i+j}|M_{ij}|
$$

Thus if the sum of the subscripts is an even number, $|C_{ij}| = |M_{ij}|$ since –1 raised to an even power is positive. If $i+j$ is equal to an odd number, $|C_{ij}| = -|M_{ij}|$, since –1 raised to an odd power is negative

•Laplace expansion and higher-order determinants

Laplace expansion is a method for evaluating determinants in terms of cofactors. It thus simplifies matters by permitting higher-order determinants to be established in terms of lower-order determinants. Laplace expansion of a third-order determinants can be expressed as

$$|A| = a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}|$$

Laplace expansion permits evaluation of a determinant along any row or column. Select a row or column with ore zeros than others simplifies evaluation of the determinant by eliminating terms.Laplace expansion also serves as the basis for evaluating determinants of orders higher than three.

•Properties of a determinant

- •Adding or subtracting any nonzero multiple of one row (or column) from another row (or column) will have no effect on the determinant.
- •Interchanging any two row or column of a matrix will change the sign, but not the absolute value, of the determinant
- •Multiplying the elements of any row or column by a constant will cause the determinant to be multiplied by the constant
- •The determinant of a triangular matrix,i.e., a matrix with zero elements everywhere above or below the principal diagonal, is equal to the product of the elements on the principal diagonal
- •The determinant of a matrix equals the determinant of its transpose: $|A| = |A'|$
- •If all the element of any row or column are zero, the determinant is zero
- •If two rows or columns are identical or proportional,i.e., linearly dependent, the determinant is zero

•Cofactor and adjoint matrices

A cofactor matrix is a matrix in which every element aij is replaced with it cofactor $|C_{ij}|$. An adjoint matrix is the transpose of a cofactor matrix thus

$$C = \begin{bmatrix} |C_{11}| & |C_{12}| & |C_{13}| \\ |C_{21}| & |C_{22}| & |C_{23}| \\ |C_{31}| & |C_{32}| & |C_{33}| \end{bmatrix} \qquad AdjA = C' = \begin{bmatrix} |C_{11}| & |C_{21}| & |C_{31}| \\ |C_{12}| & |C_{22}| & |C_{32}| \\ |C_{13}| & |C_{23}| & |C_{33}| \end{bmatrix}$$

•Inverse Matrices

An inverse matrix A^{-1}, which can be found only for a square, nonsingular matrix A is a unique matrix satisfying the relationship : $AA^{-1}=I=A^{-1}A$

Multiplying a matrix by its inverse reduces it to an identity matrix. Thus, the inverse matrix in linear algebra performs much the same function as the reciprocal in ordinary algebra. The formular for deriving the inverse is

$$A^{-1} = \frac{1}{|A|} AdjA$$

•Solving linear equations with the inverse

An inverse matrix can be used to solve matrix equations, if $A_{n\times n} X_{n\times 1} = B_{n\times 1}$ and the inverse A-1 exists, multiplication of both sides of the equation by A-1, following the laws of conformability, gives $A^{-1}{}_{n\times n} A^{n\times n} X_{n\times 1} = A^{-1}{}_{n\times n} B_{n\times 1}$, $A^{-1}A=I$. Thus, $I_{n\times n} X_{n\times 1} = A^{-1}{}_{n\times n} B_{n\times 1}$

IX=X. Therefore, $X_{n\times 1} = (A^{-1}B)_{n\times 1}$. The solution of the equation is given by the product of the inverse of the coefficient matrix A^{-1} and the column vector of constants B

Cramer's rule for matrix solutions

Cramer's rule provides a simplified method of solving a system of linear equations through the use of determinants. Cramer's rule states

$$\bar{x}_i = \frac{|A_i|}{|A|}$$

Where x_i is the *ith* unknown variable in a series of equations, $|A|$ is the determinant of the coefficient matrix, and $|A_i|$ is the determinant of a special matrix formed from the original coefficient matrix by replacing the column of coefficients of x_i with the column vector of constants

Special determinants and matrices and their use in economics

- The Jocobian

A Jocobian determinant permits testing for functional dependence, both linear and non-linear. A jocobian determinant $|J|$ is composed of all the first-order partial derivatives of a system of equations arranged in ordered sequence

$$y_1 = f_1(x_1, x_2, x_3)$$
$$y_2 = f_2(x_1, x_2, x_3)$$
$$y_3 = f_3(x_1, x_2, x_3)$$

$$|J| = \left|\frac{\partial y_1, \partial y_2, \partial y_3}{\partial x_1, \partial x_2, \partial x_3}\right| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

The elements of each row are the partial derivatives of one function y_i with respect to each of the independent variables x_1, x_2, x_3, and the elements of each column are the partial derivatives of each of the functions y_1, y_2, y_3, with respect to one of the independent variables x_j, if $|J| = 0$, the equations are functionally dependent; if $|J| \neq 0$ the equations are functionally independent

The Hessian

Given that the first-order condition $z_x=z_y=0$, are met, a sufficient condition for a multivariable function $x=f(x,y)$ to be at an optimum is

$$Z_{xx}, Z_{yy}>0 \quad \text{for a minimum}$$

$$Z_{xx},Z_{yy}<0 \quad \text{for a maximum}$$

$$Z_{xx}\, Z_{yy} >(Z_{xy})^2$$

A convenient test for this second-order condition is the Hessian. A Hessian $|H|$ is a determinant composed of all the second-order partial derivatives, with the second-order direct partials on the principal diagonal and the second-order cross partials off the principal diagonal.thus

$$|H| = \begin{vmatrix} Z_{xx} & Z_{xy} \\ Z_{xy} & Z_{yy} \end{vmatrix}$$

Where $Z_{xy}=Z_{yx}$. If the first element on the principal diagonal, the first principal minor, $|H_1| = Z_{xx}$ is positive and the second principal minor

$$|H_2| = \begin{vmatrix} Z_{xx} & Z_{xy} \\ Z_{xy} & Z_{yy} \end{vmatrix} = Z_{xx}Z_{yy} - (Z_{xy})^2 > 0$$

The second-order conditions for a minimum are met. When $|H_1|>0$ and $|H_2|>0$ the Hessian $|H|$ is called positive definite. A positive definite Hessian fulfills the second-order conditions for a minimum. If the first principal minor $|H_1| = Z_{xx} < 0$ is negative and the second principal minor

$$|H_2| = \begin{vmatrix} Z_{xx} & Z_{xy} \\ Z_{xy} & Z_{yy} \end{vmatrix} > 0$$

The second-order conditions for a maximum are met. When $|H_1|<0$ and $|H_2|>0$ the Hessian $|H|$ is called negative definite. A negative definite Hessian fulfills the second-order conditions for a maximum.